

Agenda

1. Adding student-level predictors
2. Adding class-level predictors
3. Random intercepts in R

Intercept-only model

**Partial pooling
(random effects)**

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = a_k$$

$$a_k \sim \text{Norm}(\gamma, \phi)$$

$$\sigma \sim \text{Unif}(0, 100)$$

$$\gamma \sim \text{Norm}(500, 100)$$

$$\phi \sim \text{Unif}(0, 100)$$

Accounting for student race

Number of students by race/ethnicity

<i>White</i>	4222
<i>Black</i>	2126
<i>Asian</i>	19
<i>Hispanic</i>	9
<i>Native American</i>	4
<i>Other</i>	11
<i>Total</i>	6391

Number of classes by experimental condition

<i>Small</i>	122
<i>Large</i>	114
<i>Large + Aide</i>	98
<i>Total</i>	334

Accounting for student race

But first,
some
notation:

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_1 B_i$$

β_{0k} is the
intercept for
classroom k

$$\beta_{0k} \sim \text{Norm}(\gamma_0, \phi_0)$$

Coefficient β_1 measures
difference in test scores for
Black and non-Black students

Subscripts on γ and ϕ to
remind us which coefficient
they refer to (β_{0k})

Accounting for student race

But first,
some
notation:

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_1 B_i$$

$$\beta_{0k} \sim \text{Norm}(\gamma_0, \phi_0)$$

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_1 B_i$$

$$\beta_{0k} = \gamma_0 + \eta_{0k}$$

$$\eta_{0k} \sim \text{Norm}(0, \phi_0)$$

Equivalent ways to
describe the same
distribution for β_{0k}

Accounting for student race

Full model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\begin{aligned} \mu_{ik} = & \beta_{0k} + \beta_1 \text{Black}_i + \beta_2 \text{Asian}_i \\ & + \beta_3 \text{Hispanic}_i + \beta_4 \text{NativeAmerican}_i \\ & + \beta_5 \text{Other}_i \end{aligned}$$

$$\beta_{0k} = \gamma_0 + \eta_{0k}$$

$$\eta_{0k} \sim \text{Norm}(0, \phi_0)$$

$$\beta_1, \dots, \beta_5 \sim \text{Norm}(0, 50)$$

$$\sigma \sim \text{Unif}(0, 100)$$

$$\gamma_0 \sim \text{Norm}(500, 100)$$

$$\phi_0 \sim \text{Unif}(0, 100)$$

Accounting for student race

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\begin{aligned} \mu_{ik} = & \beta_{0k} + \beta_1 \text{Black}_i + \beta_2 \text{Asian}_i \\ & + \beta_3 \text{Hispanic}_i + \beta_4 \text{NativeAmerican}_i \\ & + \beta_5 \text{Other}_i \end{aligned}$$

$$\beta_{0k} = \gamma_0 + \eta_{0k}$$

$$\eta_{0k} \sim \text{Norm}(0, \phi_0)$$

$$\beta_1, \dots, \beta_5 \sim \text{Norm}(0, 50)$$

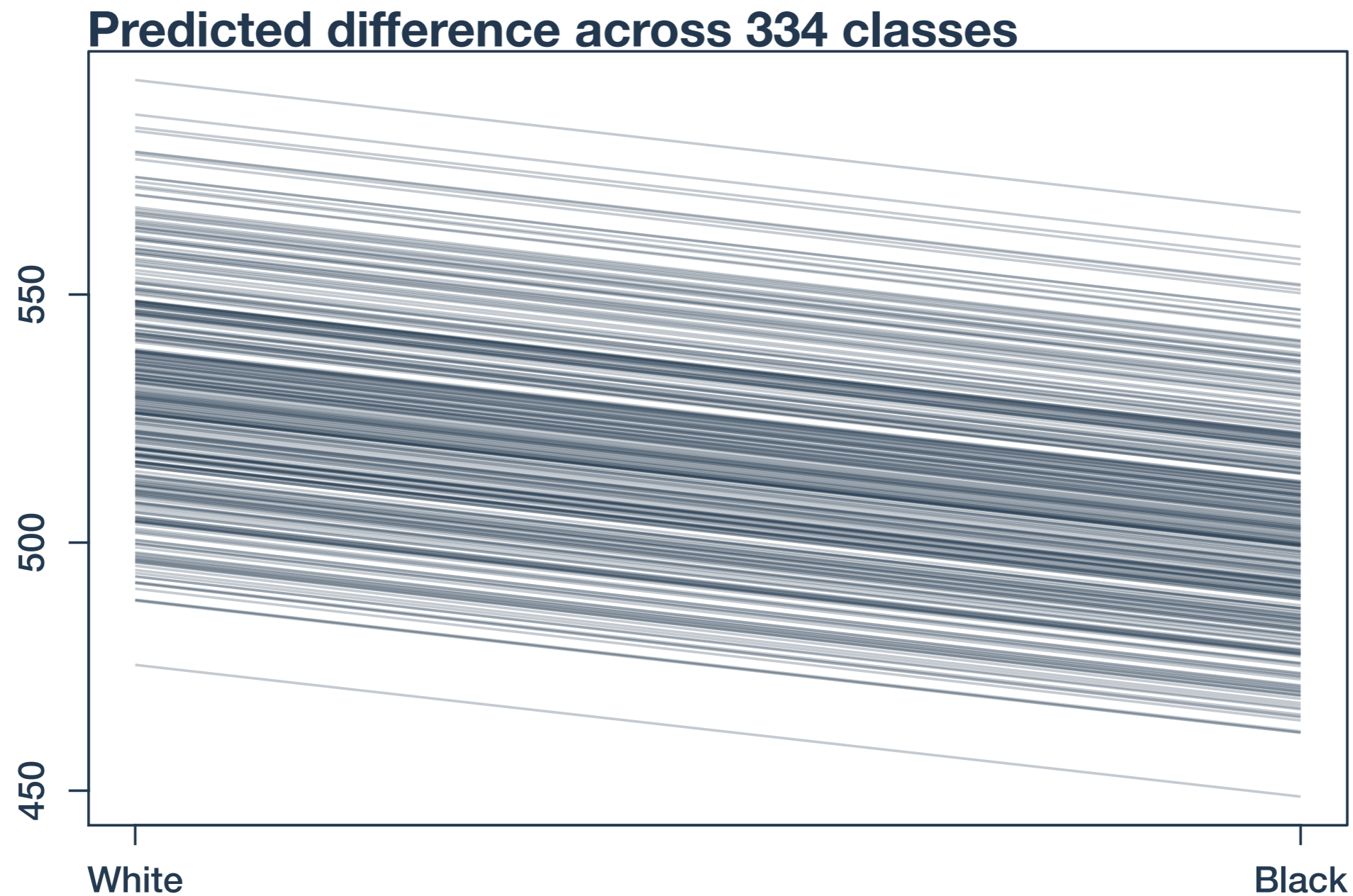
$$\sigma \sim \text{Unif}(0, 100)$$

$$\gamma_0 \sim \text{Norm}(500, 100)$$

$$\phi_0 \sim \text{Unif}(0, 100)$$

	<i>Mean</i>	<i>90% credible interval</i>	
γ_0	530.65	528.21	533.29
β_1	-26.78	-29.52	-23.17
β_2	6.70	-9.03	25.07
β_3	16.39	-10.53	41.08
β_4	-13.10	-49.60	20.18
β_5	14.81	-9.17	33.92
σ	47.03	46.37	47.75
ϕ_0	23.98	22.01	25.83

Accounting for student race



$$\gamma_0 = 530.65$$

$$\beta_1 = -26.78$$

$$\phi_0 = 23.98$$

Comparing to pooled model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\begin{aligned} \mu_{ik} = & \beta_0 + \beta_1 \text{Black}_i + \beta_2 \text{Asian}_i \\ & + \beta_3 \text{Hispanic}_i + \beta_4 \text{NativeAmerican}_i \\ & + \beta_5 \text{Other}_i \end{aligned}$$

$$\beta_0, \dots, \beta_5 \sim \text{Norm}(0, 50)$$

$$\sigma \sim \text{Unif}(0, 100)$$

	<i>Mean</i>	<i>90% credible interval</i>	
β_0	533.00	531.59	534.29
β_1	-36.86	-39.11	-34.71
β_2	12.61	-7.84	29.98
β_3	7.61	-21.58	31.06
β_4	-28.47	-66.14	8.46
β_5	13.83	-10.70	35.93
σ	52.37	51.49	53.07

Comparing to pooled model

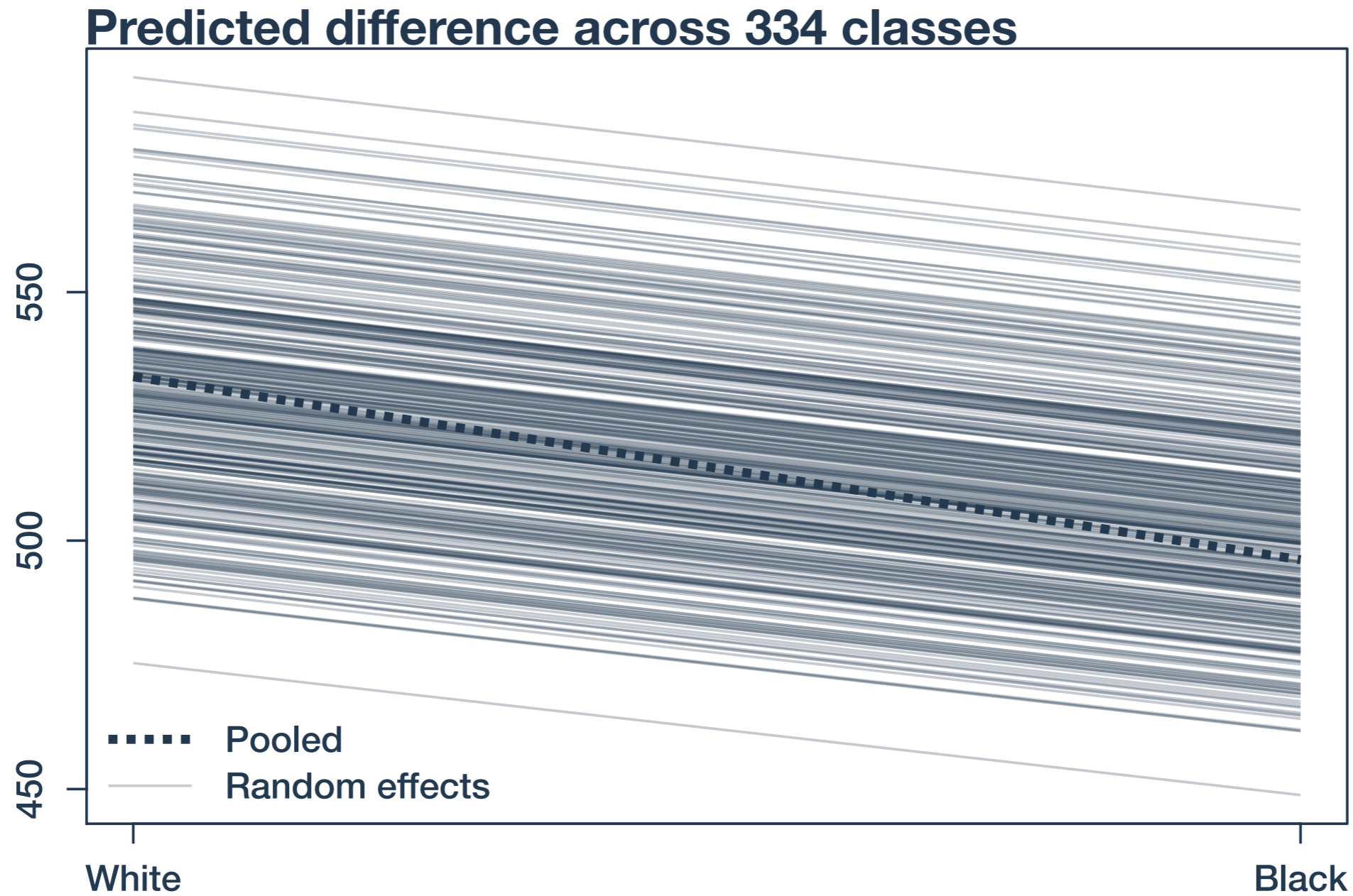
Random intercept

	<i>Mean</i>	<i>90% credible interval</i>	
γ_0	530.65	528.21	533.29
β_1	-26.78	-29.52	-23.17
β_2	6.70	-9.03	25.07
β_3	16.39	-10.53	41.08
β_4	-13.10	-49.60	20.18
β_5	14.81	-9.17	33.92
σ	47.03	46.37	47.75
ϕ_0	23.98	22.01	25.83

Pooled

	<i>Mean</i>	<i>90% credible interval</i>	
β_0	533.00	531.59	534.29
β_1	-36.86	-39.11	-34.71
β_2	12.61	-7.84	29.98
β_3	7.61	-21.58	31.06
β_4	-28.47	-66.14	8.46
β_5	13.83	-10.70	35.93
σ	52.37	51.49	53.07

Accounting for student race



$$\text{Random effects} \left| \begin{array}{l} \gamma_0 = 530.65 \\ \beta_1 = -26.78 \\ \phi_0 = 23.98 \end{array} \right.$$

$$\text{Pooled} \left| \begin{array}{l} \beta_0 = 533.00 \\ \beta_1 = -36.86 \end{array} \right.$$

Accounting for class size

**Number of
participants by
race/ethnicity**

<i>White</i>	4222
<i>Black</i>	2126
<i>Asian</i>	19
<i>Hispanic</i>	9
<i>Native American</i>	4
<i>Other</i>	11
<i>Total</i>	6391

**Number of classes
by experimental
condition**

<i>Small</i>	122
<i>Large</i>	114
<i>Large + Aide</i>	98
<i>Total</i>	334

Accounting for class size

Full model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\begin{aligned} \mu_{ik} = & \beta_{0k} + \beta_1 \text{Black}_i + \beta_2 \text{Asian}_i \\ & + \beta_3 \text{Hispanic}_i + \beta_4 \text{NativeAmerican}_i \\ & + \beta_5 \text{Other}_i \end{aligned}$$

$$\beta_{0k} = \gamma_0 + \gamma_1 \text{Small}_k + \eta_{0k}$$

$$\eta_{0k} \sim \text{Norm}(0, \phi_0)$$

$$\beta_1, \dots, \beta_5 \sim \text{Norm}(0, 50)$$

$$\sigma \sim \text{Unif}(0, 100)$$

$$\gamma_0 \sim \text{Norm}(500, 100)$$

$$\gamma_1 \sim \text{Norm}(0, 50)$$

$$\phi_0 \sim \text{Unif}(0, 100)$$

Coefficient γ_1 measures average difference in test score for classes in the “small” experimental condition.

Accounting for class size

Note For computational efficiency and other pragmatic reasons, second-level terms like γ_1 sometimes need to be implemented as first-level components.

In random-intercept models, this is a trivial matter, but it gets more complex with models we will cover soon.

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$
$$\mu_{ik} = \beta_{0k} + \beta_1 \text{Black}_i + \beta_2 \text{Asian}_i$$
$$+ \beta_3 \text{Hispanic}_i + \beta_4 \text{NativeAmerican}_i$$
$$+ \beta_5 \text{Other}_i$$

$$\beta_{0k} = \gamma_0 + \gamma_1 \text{Small}_k + \eta_{0k}$$
$$\eta_{0k} \sim \text{Norm}(0, \phi_0)$$

Think about γ_1 as a second-level variable.

Mathematically, γ_1 can be included at the first level.

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$
$$\mu_{ik} = \beta_{0k} + \gamma_1 \text{Small}_k + \beta_1 \text{Black}_i + \beta_2 \text{Asian}_i$$
$$+ \beta_3 \text{Hispanic}_i + \beta_4 \text{NativeAmerican}_i$$
$$+ \beta_5 \text{Other}_i$$

$$\beta_{0k} = \gamma_0 + \eta_{0k}$$
$$\eta_{0k} \sim \text{Norm}(0, \phi_0)$$

Accounting for class size

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\begin{aligned} \mu_{ik} = & \beta_{0k} + \beta_1 \text{Black}_i + \beta_2 \text{Asian}_i \\ & + \beta_3 \text{Hispanic}_i + \beta_4 \text{NativeAmerican}_i \\ & + \beta_5 \text{Other}_i \end{aligned}$$

$$\beta_{0k} = \gamma_0 + \gamma_1 \text{Small}_k + \eta_{0k}$$

$$\eta_{0k} \sim \text{Norm}(0, \phi_0)$$

	<i>Mean</i>	<i>90% credible interval</i>	
γ_0	526.15	523.34	529.04
γ_1	12.37	8.26	16.82
β_1	-26.70	-29.88	-23.51
β_2	7.34	-8.59	24.19
β_3	16.37	-6.66	42.92
β_4	-14.17	-48.66	23.74
β_5	14.83	-8.74	35.25
σ	47.03	46.33	47.69
ϕ_0	23.32	21.32	25.31
β_{1k}	⋮	⋮	⋮

Accounting for class size

